

BABUŠKA'S PARADOX IN A NONLINEAR BENDING-FOLDING MODEL

SÖREN BARTELS, ANDREA BONITO, PETER HORNUNG,
AND MICHAEL NEUNTEUFEL

ABSTRACT. The Babuška or plate paradox concerns the failure of convergence when a domain with curved boundary is approximated by polygonal domains in linear bending problems with simply supported boundary conditions. It can be explained via a boundary integral representation of the total Gaussian curvature that is part of the Kirchhoff–Love bending energy. It is shown that the paradox also occurs for a nonlinear bending-folding model which enforces vanishing Gaussian curvature. A simple remedy that is compatible with simplicial finite element methods to avoid incorrect convergence is devised.

1. INTRODUCTION

1.1. Babuška's paradox. A remarkable observation due to Babuška, cf. [BabPit90], is that canonical approximations of certain fourth order problems may fail to converge when curved domains are approximated using polygons. In particular, this occurs for the Kirchhoff–Love bending energy

$$I(\omega; v) = \frac{\sigma}{2} \int_{\omega} |\Delta v|^2 \, dx + \frac{1-\sigma}{2} \int_{\omega} |D^2 v|^2 \, dx$$

defined on the set of functions $v \in V(\omega) = H^2(\omega) \cap H_0^1(\omega)$ corresponding to simply supported boundary conditions. Then, the approximating functionals $I(\omega_m; \cdot)$ with a sequence of approximating polygons ω_m for ω do not converge in variational sense to $I(\omega, \cdot)$ if ω has curved boundary parts. The incorrect convergence is illustrated in Figure 1. A simple explanation follows from the relation

$$\begin{aligned} \int_{\hat{\omega}} |D^2 v|^2 \, dx &= \int_{\hat{\omega}} |\Delta v|^2 - 2 \det D^2 v \, dx \\ &= \int_{\hat{\omega}} |\Delta v|^2 \, dx - \int_{\partial \hat{\omega}} \hat{\kappa} |\partial_{\nu} v|^2 \, ds, \end{aligned}$$

which holds for Lipschitz domains $\hat{\omega} \subset \mathbb{R}^2$ whose boundaries consist of finitely many C^2 arcs whose piecewise curvature is denoted by $\hat{\kappa}$ which is positive for locally convex arcs, cf. [CoNiSw19] for a related density result. To address the variational convergence of functionals associated with

Date: December 4, 2025.

2010 Mathematics Subject Classification. 74K20 74G65 65N30.

Key words and phrases. Plate bending, domain approximation, folding, convergence.

approximating domains $\omega_m \subset \omega$ we trivially extend functions and their derivatives defined on ω_m to functions on ω by assigning the value zero in $\omega \setminus \omega_m$. Since the functionals $I(\omega_m; \cdot)$ and $I(\omega; \cdot)$ are quadratic expressions in the Hessian we have that if $(v_m)_{m \geq 0} \subset H_0^1(\omega)$ is a sequence with $v_m \in H^2(\omega_m) \cap H_0^1(\omega_m)$ and $v_m \rightharpoonup v$ in $H_0^1(\omega)$ then $I(\omega_m; v_m) \rightarrow I(\omega; v)$ implies that the trivial extensions of $D^2 v_m$ converge strongly to $D^2 v$ in $L^2(\omega)$. This leads to a contradiction since the boundary integral terms in $I(\omega_m; \cdot)$ vanish for every $m \geq 0$ but provide a nontrivial contribution to $I(\omega; \cdot)$ unless the boundary of ω is piecewise straight or the normal derivative vanishes on $\partial\omega$. Note that also distributional curvature contributions in the corner points do not improve the convergence since in those points we have $\partial_\nu v = 0$. The paradox is thus a consequence of an insufficient convergence of the approximating boundary curvatures κ_m .

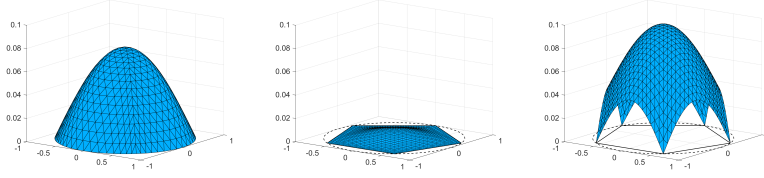


FIGURE 1. Interpolant of the exact deflection (left), approximation imposing the boundary condition along the entire boundary (middle), and approximation obtained imposing the boundary condition in the corner points (right); pictures taken from [BarTsc24].

1.2. Remedies. If domains ω_m with piecewise quadratic boundaries are used to approximate ω then the boundary curvatures converge as functions and the paradox is avoided. To avoid the failure of convergence and still permit the use of polygonal approximations it suffices to relax the boundary condition in the approximating problems by imposing it only in the corner points of ω_m and in parts where $\partial\omega$ and $\partial\omega_m$ coincide, cf. [Rann79, BarTsc24]. Figure 1 illustrates the modification. Hence, we set

$$\tilde{V}_m = \{v \in H^2(\omega_m) : v = 0 \text{ on } \partial\omega \cap \partial\omega_m\}.$$

In this case, it is straightforward to establish a Γ convergence result with respect to weak convergence in $H_0^1(\omega)$ for $\tilde{I}(\omega_m; \cdot)$ with admissible sets $\tilde{V}_m$ to $I(\omega; \cdot)$ defined on $V(\omega)$. If $v_m \rightarrow v$ in L^2 with functions $v_m \in \tilde{V}_m$ and $I(\omega_m; v_m)$ is bounded then the trivial extensions of the Hessians $D^2 v_m$ converge weakly in $L^2(\omega)$ to $D^2 v$ while suitable interpolants $\mathcal{I}_m v_m \in H_0^1(\omega_m)$ converge weakly to $v \in H_0^1(\omega)$. Hence, $v \in V$ and $I(\omega; v) \leq \liminf_{m \rightarrow \infty} I(\omega_m; v_m)$. Given a function $v \in V$, the restrictions $v_m = v|_{\omega_m}$ belong by construction to the spaces $\tilde{V}_m$ and we have that $I(\omega_m; v_m) \rightarrow I(\omega; v)$ as $m \rightarrow \infty$. We refer to [Davi02, NaSwSt11, ArnWal20] for other approximations that lead to correct convergence.

1.3. Bending and folding. Nonlinear bending models often involve an isometry condition on deformations which implies that the Frobenius norms of the Hessian and the Laplacian coincide. Recalling that their difference causes the critical boundary term in the small deflection model raises the question whether the paradox may in fact be an artifact of an oversimplistic model reduction. For deformations instead of deflections the Hessians and the Laplacians contain more information and the above argument does not apply. To show that failure of convergence also takes place in models describing large bending deformations we consider a Kirchhoff model that allows for the folding of thin elastic sheets along a given crease line $\gamma \subset \bar{\omega}$. This has applications in the construction of biomimetic devices, cf. [Saff^{et al}'19]; cf. [ConMag08] for related mathematical problems. Along the crease line only continuity is imposed while away from it nonlinear bending is measured via a piecewise Kirchhoff bending energy, i.e., following [FrJaMu02, BaBoHo22, LiuJam24] we consider the minimization of the functional

$$I(v) = \frac{1}{2} \int_{\omega \setminus \gamma} |D^2 v|^2 dx.$$

Shearing and stretching effects are inadmissible which is encoded by the condition that deformations are isometries. Incorporating also the continuity across the crease line, the set of admissible deformations is given by

$$V(\omega, \gamma) = \{v \in H^2(\omega \setminus \gamma; \mathbb{R}^3) \cap W^{1,\infty}(\omega; \mathbb{R}^3) : (\nabla v)^T (\nabla v) = I_{2 \times 2}\}.$$

For finite element discretizations it is attractive to approximate γ by a sequence of piecewise straight curves γ_m . This gives rise to the approximating energy functionals

$$I_m(v) = \frac{1}{2} \int_{\omega \setminus \gamma_m} |D^2 v|^2 dx$$

in the admissible sets

$$V(\omega, \gamma_m) = \{v \in H^2(\omega \setminus \gamma_m; \mathbb{R}^3) \cap W^{1,\infty}(\omega; \mathbb{R}^3) : (\nabla v)^T (\nabla v) = I_{2 \times 2}\}.$$

It turns out however, that the folding of isometries is impossible along polygons as this leads to singularities in the deformation gradients which prevent a correct convergence.

1.4. Approximations using slits. As in the linear setting, we relax the continuity condition by using a perforation and imposing continuity only at the vertices $c_0, c_1, \dots, c_m$ of the curves γ_m , i.e., considering admissible sets

$$\begin{aligned} \tilde{V}(\omega, \gamma_m) = \{v \in H^2(\omega \setminus \gamma_m; \mathbb{R}^3) \cap W^{1,\infty}(\omega \setminus \gamma_m; \mathbb{R}^3) : (\nabla v)^T (\nabla v) = I_{2 \times 2}, \\ v \text{ continuous in } c_0, c_1, \dots, c_m\}. \end{aligned}$$

To establish the variational convergence of approximations $\tilde{I}_m$ with admissible sets $\tilde{V}_m$ to I defined on V we define fattened crease lines $\hat{\gamma}_m$ as unions of triangles along the polygonal crease lines γ_m , so that the exact crease line is contained in the union of these triangles. Consequently, the resulting

approximating subdomains $\hat{\omega}_m^i$, $i = 1, 2$, are contained in the exact subdomains ω^i , $i = 1, 2$, cf. Figure 2. These inclusion consistencies allow for the straightforward construction of a recovery sequence but do not appear to matter in numerical simulations. To identify a limit v of a sequence (v_m) of deformations $v_m \in \tilde{V}_m$, we choose triangulations $\mathcal{T}_m$ that contain the triangles that define the fattened crease line $\hat{\gamma}_m$ and carry out a linear nodal interpolation of v_m which defines functions $\mathcal{I}_m v_m \in W^{1,\infty}(\omega)$ with interpolation error $\mathcal{I}_m v_m - v_m$ that converges strongly to zero in $H^1(\omega'; \mathbb{R}^3)$ for domains ω' with positive distance to the exact crease line γ .

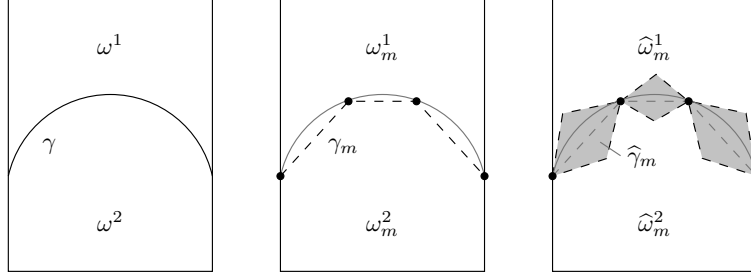


FIGURE 2. Crease line γ (left), polygonal approximation γ_m (middle), and fattened polygonal crease line $\hat{\gamma}_m$ (right). In each case the lines define partitions of the domain ω . For the fattened crease line $\hat{\gamma}_m$ we have $\hat{\omega}_m^i \subset \omega^i$, $i = 1, 2$.

We remark that our arguments to establish the convergence of approximations also apply when a curved boundary part instead of a crease line is approximated by a polygon to avoid possible incorrect convergence.

1.5. Confirmation by experiments. The locking effect introduced by approximating curved crease lines by polygonal ones and still imposing continuity is confirmed by real and numerical experiments. Figure 3 shows an experiment with a thin elastic sheet and curved crease line. When it is approximated using a simple polygonal curve and continuity is imposed along the entire line, stress concentrations occur at the vertices which are accompanied by piecewise flat deformations in a neighborhood of the crease line. Introducing slits along the segments relaxes the situation and the experiment indicates correct convergence.

The real experiments can be simulated by means of a finite element discretization of the bending-folding model. We use a mixed formulation to define a discrete second fundamental form based on the Hellan–Herrman–Johnson element. This element has the particular feature that it can be used on curved elements, in contrast to many other finite element methods developed for fourth order problems. Details of the numerical method are provided in Section 4. Figure 4 shows the results of canonical discretizations using curved crease-line approximations, piecewise straight discrete crease lines, and straight crease lines imposing continuity only at the vertices. The

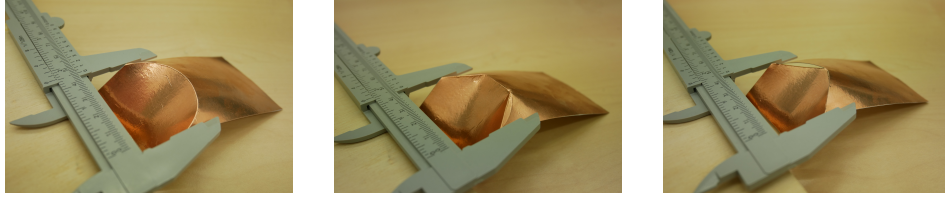


FIGURE 3. Bending of an elastic plate via compressing the plate at the end-points of a crease line. Curved crease line (left), singularities occur when a polygonal approximation is used (middle), these disappear if slits are introduced along the straight segments (right).

deformations obtained with curved and slit approximations are nearly indistinguishable while the one with continuity and straight segments leads to a reduced deformation and stress concentrations at the vertices.

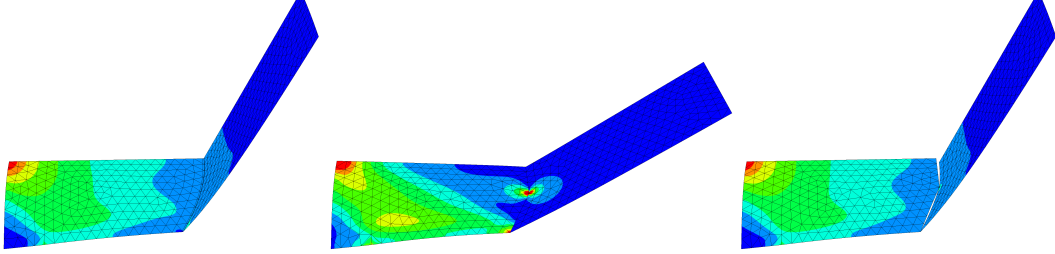


FIGURE 4. Deformations and energy densities as coloring in the simulation of folding and bending experiments using the symmetry of the problem along the long midline. A correct discrete deformation is obtained for a curved approximation of the crease line (left), while the polygonal approximation leads to flatter pieces and a singularity (middle), introducing discontinuities along the straight segments provides another correct approximation (right).

1.6. Outline. The article is organized as follows. A variational convergence result for approximations imposing continuity only in vertices is stated in Section 2. In Section 3 we recall an angle-curvature relation for folded isometries which implies that isometries are flat along straight segments of crease lines. This implies the nonexistence of nontrivially folded isometries for polygonal crease lines that are continuously differentiable in the subdomains. Additional results from numerical experiments are provided in Section 4.

2. DISCONTINUOUS APPROXIMATIONS

In this section we verify the convergence of the approximate minimization problems which only impose continuity of deformations in the vertices of a polygonal crease line approximation. To avoid assumptions about the extension of isometries we use a fattened crease line, cf. the right plot in Figure 2. This ensures that subdomains of approximating problems are subsets of the subdomains of the continuous problem which allows for a straightforward identification of recovery sequences.

Given a piecewise C^1 curve γ parametrized by a function $b \in W^{1,\infty}(\alpha, \beta; \mathbb{R}^2)$ with $|b'| = 1$ which partitions ω into two disjoint Lipschitz domains ω_1 and ω_2 we consider the minimization of the functional

$$I(v) = \frac{1}{2} \int_{\omega \setminus \gamma} |D^2 v|^2 dx,$$

defined on the set of deformations that are isometric and piecewise H^2 , i.e., on the set

$$V = \{v \in H^2(\omega \setminus \gamma; \mathbb{R}^3) \cap W^{1,\infty}(\omega; \mathbb{R}^3) : (\nabla v)^T(\nabla v) = I_{2 \times 2}\}.$$

We note that $V \neq \emptyset$ holds since unfolded isometries are contained in V . For a sequence of polygonal approximations (γ_m) of γ that are obtained by linear interpolations b_m of b we choose matching, shape-regular triangulations $(\mathcal{T}_m)$ of ω with vanishing maximal mesh-size as $m \rightarrow \infty$. We then consider the fattened crease lines $\hat{\gamma}_m$ obtained as the union of triangles $T \subset \bar{\omega}$ for which one side belongs to γ_m . We assume that the triangulation is sufficiently fine so that the exact crease line γ is contained in $\hat{\gamma}_m$, cf. Figure 2. The fattened crease line gives rise to a disjoint partitioning (up to boundary points)

$$\omega = \hat{\omega}_{m,1} \cup \hat{\gamma}_m \cup \hat{\omega}_{m,2}$$

such that $\hat{\omega}_{m,\ell} \subset \omega_\ell$ and $\gamma \subset \hat{\gamma}_m$. We thus consider the functionals

$$\tilde{I}_m(v) = \frac{1}{2} \int_{\omega \setminus \hat{\gamma}_m} |D^2 v|^2 dx$$

defined on the set of isometric deformations that are continuous in the vertices $c_0, c_1, \dots, c_m$ of γ_m , and H^2 -regular in the subdomains, i.e., on the set

$$\begin{aligned} \tilde{V}_m = \{v \in H^2(\omega \setminus \hat{\gamma}_m; \mathbb{R}^3) : (\nabla v)^T(\nabla v) = I_{2 \times 2}, \\ v \text{ continuous in } c_0, c_1, \dots, c_m\}. \end{aligned}$$

We remark that we have $\tilde{V}_m \subset W^{1,\infty}(\omega \setminus \hat{\gamma}_m; \mathbb{R}^3)$. We formally extend the functionals by $+\infty$ to deformations $v \in L^2(\omega; \mathbb{R}^3) \setminus \tilde{V}_m$. Functions in $\tilde{V}_m$ and their derivatives are identified with their trivial extensions to ω throughout the following.

A compactness result follows by an approximate extension of functions $v_m \in \tilde{V}_m$ to functions $\tilde{v}_m \in W^{1,\infty}(\omega; \mathbb{R}^3)$ via a nodal interpolation of v_m in the triangulations of ω that contain the fattened crease lines $\hat{\gamma}_m$.

Proposition 2.1 (Compactness). *Let $(v_m) \subset L^2(\omega; \mathbb{R}^3)$ be such that $v_m \in \tilde{V}_m$ and $\tilde{I}_m(v_m) \leq c$ for all $m \in \mathbb{N}$. Then there exists a sequence $(\tilde{v}_m) \subset W^{1,\infty}(\omega, \mathbb{R}^3)$ such that $\|\nabla \tilde{v}_m\|_{L^\infty(\omega)} \leq c'$ and $\tilde{v}_m - v_m \rightarrow 0$ in $H^1(\omega'; \mathbb{R}^3)$ for every ω' that is compactly contained in $\omega \setminus \gamma$. If $v \in W^{1,\infty}(\omega; \mathbb{R}^3)$ is a weak- $\star$ accumulation point of $(\tilde{v}_m)$ then we have $v \in V$ and D^2v is a weak accumulation point in $L^2(\omega; \mathbb{R}^{3 \times 2 \times 2})$ of (D^2v_m) .*

Proof. Let $(\mathcal{T}_m)$ be a sequence of regular triangulations of ω that match the curves (γ_m) . Each $\mathcal{T}_m$ contains the triangles that define the fattened crease line $\hat{\gamma}_m$ such that the maximal diameter h_m tends to zero. Since all nodes of $\mathcal{T}_m$ belong to the closure of $\omega \setminus \hat{\gamma}_m$ the nodal interpolant $\mathcal{I}_m v_m$ is well defined for every $v_m \in \tilde{V}_m$. Moreover, $\|\nabla \mathcal{I}_m v_m\|_{L^\infty(\omega)} \leq c_m \|\nabla v_m\|_{L^\infty(\omega \setminus \hat{\gamma}_m)} \leq c_m$. The uniform shape regularity implies that c_m is uniformly bounded. Setting $\tilde{v}_m = \mathcal{I}_m v_m$ thus proves the first part.

Let $v \in W^{1,\infty}(\omega; \mathbb{R}^3)$ be a weak- $\star$ accumulation point of $(\tilde{v}_m)$ and note that the (piecewise) Hessians D^2v_m have a weak accumulation point X in $L^2(\omega; \mathbb{R}^{3 \times 2 \times 2})$ since $\tilde{I}_m(v_m)$ is bounded. Since the differences $v_m - \tilde{v}_m$ converge strongly to zero in $H^1(\omega'; \mathbb{R}^3)$ for every set ω' that is compactly contained in $\omega \setminus \gamma$, it follows that $v \in H^2(\omega \setminus \gamma; \mathbb{R}^3)$ with $D^2v = X$ and $(\nabla v)^\top (\nabla v) = I_{2 \times 2}$, i.e., $v \in V$. $\square$

The convergence result is an immediate consequence of the construction of the approximations.

Proposition 2.2 (Gamma convergence). *(i) If $(v_m) \subset L^2(\omega; \mathbb{R}^3)$ is such that $v_m \in \tilde{V}_m$ and $\tilde{I}_m(v_m) \leq c$ then there exists $v \in L^2(\omega; \mathbb{R}^3)$ such that $v_m \rightarrow v$ in $L^2(\omega; \mathbb{R}^3)$ (up to the selection of a subsequence) and $v \in V$ with $I(v) \leq \liminf_{m \rightarrow \infty} \tilde{I}_m(v_m)$.
(ii) If $v \in V$ then there exists a sequence (v_m) such that $v_m \in V_m$ for all m , $\lim_{m \rightarrow \infty} v_m = v$ in $L^2(\omega; \mathbb{R}^3)$, and $I(v) = \lim_{m \rightarrow \infty} I_m(v_m)$.*

Proof. (i) Using Proposition 2.1 we obtain uniformly bounded functions $\tilde{v}_m \in W^{1,\infty}(\omega; \mathbb{R}^3)$. After extraction of a subsequence, this provides a limit $v \in W^{1,\infty}(\omega; \mathbb{R}^3)$ with $\tilde{v}_m \rightharpoonup^* v$ in $W^{1,\infty}(\omega; \mathbb{R}^3)$. Proposition 2.1 guarantees that $v \in V$ and $I(v) \leq \liminf_{m \rightarrow \infty} I_m(v_m)$. This proves the first statement. (ii) Given $v \in V$ we note that the restrictions $v_m = v|_{\hat{\omega}_m}$ satisfy $v_m \in V_m$. Their trivial extensions converge in L^2 to v , and the energies converge, since $|\omega_\ell \setminus \hat{\omega}_{m,\ell}| \rightarrow 0$ as $m \rightarrow \infty$. $\square$

The main implication of the convergence result concerns the accumulation of almost-minimizers at minimizers.

Corollary 2.3 (Convergence of almost-minimizers). *Assume that $(u_m) \subset L^2(\omega; \mathbb{R}^3)$ is such that $u_m \in \tilde{V}_m$ and $\tilde{I}_m(u_m) \leq \min_{v \in \tilde{V}_m} I(v) + \delta_m$ for a sequence of positive numbers $\delta_m \rightarrow 0$. Then there exists a minimizer $u \in V$ for I such that $u_m \rightarrow u$ and $D^2u_m \rightharpoonup D^2u$ weakly in $L^2(\omega; \mathbb{R}^3)$.*

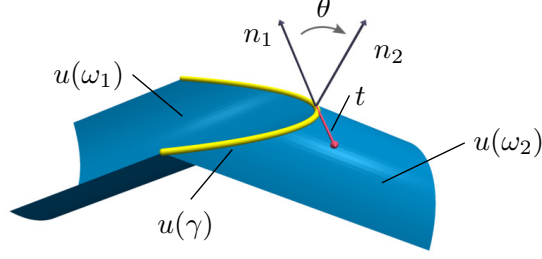


FIGURE 5. Folding angle between the jumping normals along a crease which partitions the deformed sheet into parts of opposite curvatures and defines two isometries of the subdomains. The induced Darboux frames specify curvature and torsion quantities for the deformed folding arc $u(\gamma)$.

3. ANGLE-CURVATURE RELATIONS AND NONEXISTENCE

3.1. Folding angle. We consider a connected folding arc (segment) γ that is parametrized by the embedded arclength curve $b \in W^{2,\infty}(\alpha, \beta; \mathbb{R}^2)$ and let $u \in H^2(\omega \setminus \gamma; \mathbb{R}^3)$ be a folded isometry, i.e.,

$$u \in V = \{v \in H^2(\omega \setminus \gamma; \mathbb{R}^3) \cap W^{1,\infty}(\omega; \mathbb{R}^3) : (\nabla v)^\top (\nabla v) = I_{2 \times 2}\}.$$

The curve γ is assumed to partition ω into two subdomains ω_1 and ω_2 . We further assume that the restrictions $u_\ell = u|_{\omega_\ell}$, $\ell = 1, 2$, can be extended as H^2 isometries to open neighborhoods of $\bar{\omega}_\ell$. The mapping $u \circ b$ provides an arclength parametrization of the deformed folding arc with unit tangent vector

$$t = \gamma' = (Du \circ b)b',$$

which is the same for both u_ℓ , $\ell = 1, 2$, cf. Figure 5.

We let $n_\ell = \partial_1 u_\ell \times \partial_2 u_\ell : \bar{\omega}_\ell \rightarrow \mathbb{R}^3$, $\ell = 1, 2$, denote the continuous unit normals to the deformed adjacent surfaces in a neighborhood of γ . Along γ we identify these and other quantities with mappings defined on the interval $[\alpha, \beta]$ via, e.g., $n_\ell(s) = n_\ell(b(s))$. By the extensibility assumption and the regularity result [Kirc01, MulPak05] we have that $n_\ell \in C([\alpha, \beta]; \mathbb{R}^3)$ and

$$(1) \quad n_2 = R(\theta, t)n_1,$$

where $R(\theta, t)$ is the rotation about t by the angle θ , which satisfies

$$(2) \quad \cos \theta = n_1 \cdot n_2.$$

Choosing conormal vectors $m_\ell = n_\ell \times t$, $\ell = 1, 2$, that are tangential to the surfaces and normal to the folding curve, we consider the Darboux frames

$$r_\ell = [t, m_\ell, n_\ell],$$

for $\ell = 1, 2$. The second and third column vectors of the frames are related via

$$\begin{aligned} m_2 &= \cos(\theta)m_1 + \sin(\theta)n_1, \\ n_2 &= -\sin(\theta)m_1 + \cos(\theta)n_1. \end{aligned}$$

The frames give rise to the geodesic and normal curvatures

$$\kappa_\ell = t' \cdot m_\ell, \quad \mu_\ell = t' \cdot n_\ell,$$

and the geodesic torsions

$$\tau_\ell = (m_\ell)' \cdot n_\ell.$$

Since the isometric deformations u_ℓ preserve intrinsic quantities we have that

$$\kappa = \kappa_1 = \kappa_2.$$

The combination of this identity with the equations for m_2 and n_2 implies that we have

$$\kappa = t' \cdot m_2 = t' \cdot (\cos(\theta)m_1 + \sin(\theta)n_1) = \cos(\theta)\kappa + \sin(\theta)\mu_1,$$

i.e.,

$$(1 - \cos(\theta))\kappa = \sin(\theta)\mu_1.$$

Analogously, using that $m_1 = \cos(\theta)m_2 - \sin(\theta)n_2$, we find that

$$(1 - \cos(\theta))\kappa = -\sin(\theta)\mu_2.$$

Incorporating the trigonometric identities $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$ and $\sin(2\alpha) = 2\cos(\alpha)\sin(\alpha)$ we deduce that

$$2\sin^2\left(\frac{\theta}{2}\right)\kappa = \pm 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\mu_\ell.$$

For the geodesic torsions we similarly derive, using the orthogonality of the column vectors of r_ℓ , that

$$\begin{aligned} \tau_2 &= (m_2)' \cdot n_2 = (\cos(\theta)m_1 + \sin(\theta)n_1)' \cdot (-\sin(\theta)m_1 + \cos(\theta)n_1) \\ &= \theta'(-\sin(\theta)m_1 + \cos(\theta)n_1) \cdot (-\sin(\theta)m_1 + \cos(\theta)n_1) \\ &\quad + \left(\cos(\theta)(m_1)' + \sin(\theta)(n_1)'\right) \cdot (-\sin(\theta)m_1 + \cos(\theta)n_1) \\ &= \theta'(\sin^2(\theta) + \cos^2(\theta)) + \cos^2(\theta)(m_1)' \cdot n_1 - \sin^2(\theta)(n_1)' \cdot m_1 \\ &= \theta' + \tau_1. \end{aligned}$$

The calculations imply the following result from [Horn23-pre], which extends observations from [DunDun82].

Proposition 3.1 (Folding angle, [Horn23-pre]). *Let $u \in V$ and let $b \in W^{2,\infty}(\alpha, \beta; \mathbb{R}^2)$ be an arclength parametrization for γ . Assume that the restrictions of u to the subdomains ω_ℓ can be extended as H^2 isometries to open neighborhoods of $\overline{\omega}_\ell$ for $\ell = 1, 2$. There exists a well defined folding angle $\theta \in C([\alpha, \beta])$ satisfying (1) such that*

$$(3) \quad \kappa \sin\left(\frac{\theta}{2}\right) = \widehat{\mu} \cos\left(\frac{\theta}{2}\right),$$

with $\hat{\mu} = \mu_1$, wherever the surface is folded, i.e., $\theta \notin 2\pi\mathbb{Z}$. The induced normal curvatures and geodesic torsions are related via $\mu_2 = -\mu_1$ wherever $\theta \notin 2\pi\mathbb{Z}$, and $\tau_2 = \tau_1 + \theta'$.

The identities imply that, e.g., if $\kappa = 0$ then $\theta \in \pi\mathbb{Z}$, i.e., u is unfolded or folded back, or $\hat{\mu} = 0$. In the latter case it can be shown that the folding angle θ is constant, cf. [Horn11] for details. Whenever $\kappa \neq 0$ we have that the folding angle is zero or uniquely defined via $\theta = 2 \arctan(\hat{\mu}/\kappa)$.

3.2. Failure of convergence. The Babuška paradox arises in the context of the nonlinear bending-folding model via a nonexistence result of folded H^2 -isometries for polygonal crease lines.

Proposition 3.2 (Nonexistence). *Let $\gamma_a, \gamma_b \subset \mathbb{R}^2$ be nondegenerate disjoint open line segments with a common endpoint x_C . Let $\omega \subset \mathbb{R}^2$ be simply connected and such that $\gamma = \gamma_a \cup \gamma_b \cup \{x_C\}$ is contained in ω and both endpoints of γ belong to $\partial\omega$, cf. Figure 6. Denote the connected components of $\omega \setminus \gamma$ by ω_1 and ω_2 . Let $u_\ell \in C^1(\bar{\omega}_\ell; \mathbb{R}^3)$ and assume that $u_1 = u_2$ on γ . Then, we have that:*

- (i) *If γ_a, γ_b are not parallel, then $Du_1(x_C) = Du_2(x_C)$.*
- (ii) *If, furthermore, each u_ℓ is the restriction of an H^2 isometric immersion $\tilde{u}_\ell$ defined on an open neighbourhood of $\gamma_a \cup \gamma_b$, then $Du_1 = Du_2$ along γ .*

Proof. (i) Since the tangential derivatives of u_1 and u_2 coincide along the segments γ_a and γ_b we have that $Du_1(x_C)\tau_a = Du_2(x_C)\tau_a$ as well as $Du_1(x_C)\tau_b = Du_2(x_C)\tau_b$ with the linearly independent tangent vectors $\tau_a, \tau_b \in \mathbb{R}^2$ so that $Du_1(x_C) = Du_2(x_C)$.

(ii) Applying [Horn23-pre] to the Darboux frames associated with the immersions $\tilde{u}_\ell$ implies that the folding angle is constant on γ_a . The same is true on γ_b . Hence, by continuity of the normals along γ , the resulting continuity of θ by (2), and (i), the folding angle must be zero on all of γ . This implies that the normals n_1 and n_2 coincide along γ and hence that the deformation gradients are identical on γ . $\square$

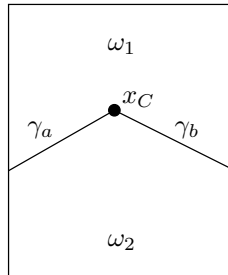


FIGURE 6. Polygonal crease $\gamma = \gamma_1 \cup \gamma_2$ line with vertex x_C

Remark 3.3. *The proposition implies that also accumulation points of piecewise C^1 regular isometric deformations in $V(\gamma_m)$ are unfolded for a sequence of polygonal crease lines. If curved segments are used then the folding angle vanishes at the vertices but may be different from zero between vertices, and the numerical experiment illustrated in the left plot of Figure 4 indicates correct convergence.*

4. NUMERICAL EXPERIMENTS

We describe in this section the numerical method that leads to the approximations shown in Figure 4. For related numerical methods to approximate isometric deformations we refer the reader to [Bart15-book, RuSiSm22, BoGuMo23, BGNy23].

4.1. Experimental setting. We consider the rectangular domain $\omega = (0, 2) \times (-1/2, 1/2)$ and define a crease line as the intersection of $\bar{\omega}$ with a circle of unit radius, i.e., $\gamma = \bar{\omega} \cap \partial B_1(0)$. The devised numerical methods impose different continuity conditions along γ and its approximations. Furthermore, we impose that no bending moments are transferred while no continuity is imposed on the deformation gradient. We impose compressive boundary conditions along the opposite boundary parts $\gamma_D = [0, 0.2] \times \{\pm 0.5\}$ in one of the subdomains, cf. Figure 7. The boundary conditions are continuously increased via a pseudo time $t \in [0, 1]$, i.e., we set

$$u_D(t, x, y) = [x, y, 0]^T + [0, -(1/10)y(1-x)t, 0]^T.$$

To guarantee that the sheet bends upwards we include a uniform vertical force that is zero at $t = 1$, i.e.,

$$f(t) = [0, 0, (2/5)(1-t^2)]^T.$$

We reduce the computational effort by exploiting the symmetry of the setting and only discretize the subdomain $\omega' = (0, 2) \times (-1/2, 0)$ imposing appropriate boundary conditions along the symmetry axis $\gamma_{\text{sym}} = [0, 2] \times \{0\}$. We use the Young's modulus $E = 10$ and a vanishing Poisson ratio which is compatible with the isometry condition. We consider three different treatments of the crease line and the continuity condition:

- (S1) The crease line is isoparametrically resolved by the numerical method and continuity is imposed along the entire crease line.
- (S2) The crease line is approximated by a polygonal curve γ_ℓ on which continuity is imposed.
- (S3) The crease is approximated by a polygonal curve γ_ℓ and continuity is imposed at the vertices of γ_ℓ .

These settings are illustrated in Figure 7. In view of our theoretical results we expect the formation of certain singularities at the vertices of the polygonal crease line in setting (S2) and correct approximations in settings (S1) and (S3). These effects are confirmed by the results of simulations shown in Figure 4. A zoom towards the simulation results in neighborhoods of

the different discrete crease lines is shown in Figure 8, which also shows the effect of the geometric refinement in case of the crease line approximation (S1).

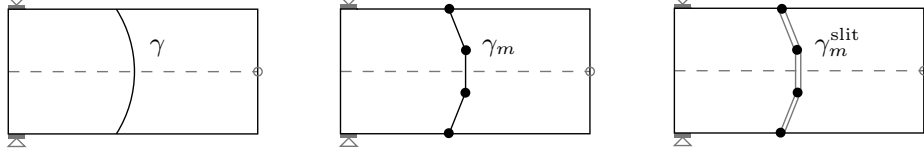


FIGURE 7. Experimental setting with boundary conditions (arrows), symmetry axis (dashed), boundary point x_P (gray dot), and crease line treatments (S1)-(S3) from left to right.

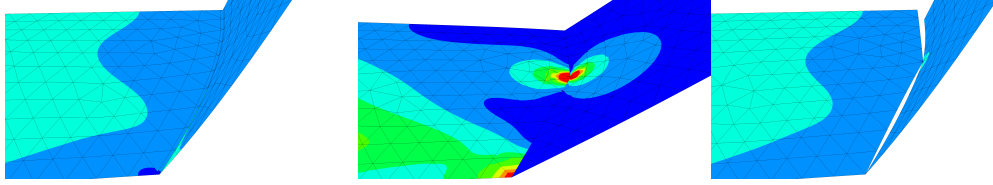


FIGURE 8. Zoom towards discrete crease lines for the folding and bending experiments corresponding to the full deformations shown in Figure 4 with crease line approximations (S1), (S2), and (S3) (from left to right) and geometric refinements towards the entire crease line in case of (S1) and towards the corner points in case of (S2) and (S3).

4.2. Saddle-point formulation. We reformulate the variational formulation of the bending-folding model as a saddle-point problem in order to use the methods devised for Koiter shells in [NS19, NS24]. We note that for isometric deformations we have the relation $|D^2v| = |\mathcal{H}_n|$ for the corresponding Frobenius norms and the second fundamental form given by

$$\mathcal{H}_n = -(\nabla v)^\top \nabla n, \quad n = \frac{\partial_x v \times \partial_y v}{|\partial_x v \times \partial_y v|},$$

where n is a unit normal vector on the deformed surface. We incorporate the second fundamental form in terms of a mixed formulation by introducing the bending moment tensor as the energetic conjugate to the curvature tensor, i.e.,

$$\mathbf{m} = -\frac{E}{12}(\nabla v)^\top \nabla n.$$

The resulting Hellinger–Reissner two-field formulation includes the isometry condition via a penalty term and is given by the Lagrange functional

$$\mathcal{L}(v, \mathbf{m}) = \int_{\omega} \alpha |(\nabla v)^\top (\nabla v) - I_{2 \times 2}|^2 - \frac{6}{E} |\mathbf{m}|^2 + ((\nabla v)^\top \nabla n) : \mathbf{m} - f \cdot v \, dx.$$

We thus aim at approximating a saddle-point $\min_v \max_{\mathbf{m}} \mathcal{L}(v, \mathbf{m})$ imposing the boundary conditions

$$v = u_D \text{ on } \gamma_D, \quad \mathbf{m}_{\nu\nu} = 0 \text{ on } \partial\omega, \quad v_y = 0 \text{ on } \gamma_{\text{sym}},$$

where $\mathbf{m}_{\nu\nu} = (\mathbf{m}\nu) \cdot \nu$, and the different continuity conditions on the crease line γ of settings (S1)-(S3) together with the condition $\mathbf{m}_{\nu\nu} = 0$ on γ or γ_ℓ for the traces from both sides with a unit normal ν along γ .

4.3. Hellan–Herrmann–Johnson method. We let $\mathcal{T}_h$ be a triangulation of ω , which contains polynomially curved elements in setting (S1). On interelement boundaries we consider the clockwise oriented unit tangent vector t and let ν be an outward unit normal. The set of edges of elements is denoted by $\mathcal{E}_h$. The jump of an elementwise continuous quantity w over an inner edge $E = T_+ \cap T_- \in \mathcal{E}_h$ with a fixed unit normal pointing from T_- into T_+ is defined as

$$[[w]]|_E = w|_{T_+} - w|_{T_-},$$

for boundary edges we simply set $[[w]]|_E = w|_E$. The deformation and bending moment fields are discretized by Lagrangian and Hellan–Herrmann–Johnson [Hel67, Her67, Joh73, Com89, Walk22] finite elements, respectively,

$$\begin{aligned} \mathcal{U}^k &= \{v \in C^0(\omega, \mathbb{R}^3) : v|_T \in \tilde{\mathcal{P}}_k(T, \mathbb{R}^3) \text{ for all } T \in \mathcal{T}_h\}, \\ \mathcal{M}^k &= \{\mathbf{m} \in L^2(\omega, \mathbb{R}_{\text{sym}}^{2 \times 2}) : \mathbf{m}|_T \in \tilde{\mathcal{P}}_k(T, \mathbb{R}_{\text{sym}}^{2 \times 2}) \text{ for all } T \in \mathcal{T}_h \text{ and} \\ &\quad [[\mathbf{m}_{\nu\nu}]]_E = 0 \text{ for all } E \in \mathcal{E}_h\}, \end{aligned}$$

where $\tilde{\mathcal{P}}_k(T)$ denotes the set of functions that are (isoparametrically) transformed polynomials. The Hellan–Herrman–Johnson stress elements satisfy the continuity of the normal-normal component of the bending stress by construction. For the normal-tangential stress component, and thus the physical normal continuity of the stress, this follows in weak sense by the discretization method. This treatment allows for a simple construction of stress elements that are required to be both symmetric and normal continuous. We use the strategies developed in [NS19, NSS23, NS24] to obtain an approximation of the Lagrange functional that can be applied to deformations that are merely continuous. Despite the elementwise second fundamental form $\nabla_h n$, which is discontinuous over sides of elements, the angle $\arccos(n^+ \cdot n^-)$ of the jump of the normal vectors across elements is considered at the edges. For a discrete pair $(v, \mathbf{m}) \in \mathcal{U}^k \times \mathcal{M}^{k-1}$ the Lagrange functional is defined via

$$\begin{aligned} \mathcal{L}^{\text{HHJ}}(v, \mathbf{m}) &= \int_{\cup \mathcal{T}_h} \alpha |(\nabla v)^\top (\nabla v) - I_{2 \times 2}|^2 - \frac{6}{E} |\mathbf{m}|^2 - \mathcal{H}_n : \mathbf{m} \, dx \\ &\quad - \int_{\omega} f \cdot v \, dx + \int_{\cup \mathcal{E}_h} \arccos(n^+ \cdot n^-) \mathbf{m}_{\nu\nu} \, ds, \end{aligned}$$

with the elementwise defined second fundamental form $\mathcal{H}_n = \sum_{i=1}^3 \nabla_h^2 v_i n_i$. The boundary conditions on γ_D are also approximated using a penalty term. This allows us to adaptively increase the penalty parameters to enforce the boundary condition and isometry constraint in the first load-step. This additional control over the constraints improves the convergence of the employed Newton's method. For further implementation details we refer to [NS19, NS24]. We remark that using the norm of the Hessian as in [Wal2024] instead of the second fundamental form introduces a simpler structure but leads to additional equations. Corresponding experimental results were nearly identical.

4.4. Numerical results. We use uniform mesh refinements and additional local geometric refinements with refinement factor 0.125 towards γ in setting (S1) and the interior vertices in case of (S2) and (S3). The reported results correspond to cubic polynomials for the deformation, i.e., we always set $k = 3$, and a fixed polygonal crease line γ_ℓ with four vertices in the full domain $\bar{\omega}$. To compare the experimental results for the different approximations of the crease line, we plotted in Figure 9 the vertical deflection $u_h \cdot e_3$ at the boundary point $x_p = (2, 0)$. We observe a reduced deflection in setting (S2) in comparison to settings (S1) and (S3), confirming the expected locking effect if continuity is imposed along a polygonal crease line. The failure of incorrect convergence is also visible in the different norms of the bending moment $\mathbf{m}$ shown in Tables 1 and 2. While the results for settings (S1) and (S3) nearly coincide, an incorrect L^2 norm and a rapidly growing L^{64} norm, which we use to approximate the L^∞ norm, are obtained. The singular effect in setting (S2) becomes worse when a second geometric refinement is carried out, i.e., we obtained $\|\mathbf{m}\|_{L^{64}} = 923.670, 1644.595, 2430.662$ for $h = 0.2, 0.1, 0.05$, respectively.

In the experiments we used the (final) penalty parameter $\alpha = 10^6$ for both, the constraint approximation and enforcement of the compressive boundary condition. We used 30 uniform load-steps for the pseudo time $t \in [0, 1]$. To improve the convergence behavior of Newton's method we use a damped version with parameter $\eta = 0.05$. Further, for the first load-step we used an internal loop to gradually reduce the violation of the isometry and deformation boundary constraints by the function $\sqrt{\beta}$ with $\beta \in [0, 1]$ to ensure that the first intermediate configuration is reached. We used 5 uniform internal steps for this purpose. We stopped Newton's method when a residual of 10^{-6} was reached.

Acknowledgments. The authors SB and PH acknowledge support by the DFG via the priority programme SPP 2256 *Variational Methods for Predicting Complex Phenomena in Engineering Structures and Materials* (441528968 (BA 2268/7-2, HO 4697/2-2)). The author AB is partially supported by NSF grant DMS-2409807.

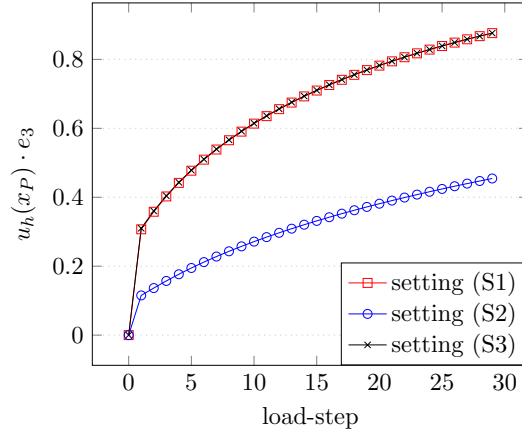


FIGURE 9. Evolution of vertical deflections during the pseudo time-stepping at $x_p = (2, 0)$ for crease line approximations of settings (S1)-(S3) and triangulations with mesh size $h = 0.05$.

h	$\ \mathbf{m}\ _{L^2}$	$\ \mathbf{m}\ _{L^{64}}$	$\ \mathbf{m}\ _{L^2}$	$\ \mathbf{m}\ _{L^{64}}$	$\ \mathbf{m}\ _{L^2}$	$\ \mathbf{m}\ _{L^{64}}$
0.2	6.841	22.173	7.417	27.226	6.838	21.938
0.1	6.832	22.042	7.457	50.182	6.827	21.704
0.05	6.832	22.512	7.503	95.972	6.828	21.714

TABLE 1. L^p norms of the bending moment tensor for uniform mesh refinement in settings (S1)-(S3) from left to right.

h	$\ \mathbf{m}\ _{L^2}$	$\ \mathbf{m}\ _{L^{64}}$	$\ \mathbf{m}\ _{L^2}$	$\ \mathbf{m}\ _{L^{64}}$	$\ \mathbf{m}\ _{L^2}$	$\ \mathbf{m}\ _{L^{64}}$
0.2	6.841	22.188	7.604	161.623	6.838	21.936
0.1	6.832	22.043	7.582	307.049	6.827	21.703
0.05	6.832	22.070	7.614	589.891	6.828	21.714

TABLE 2. L^p norms of the bending moment tensor for uniform mesh refinement with one geometric refinement in settings (S1)-(S3) from left to right.

REFERENCES

ABTEILUNG FÜR ANGEWANDTE MATHEMATIK, ALBERT-LUDWIGS-UNIVERSITÄT FREIBURG,
HERMANN-HERDER-STR. 10, 79104 FREIBURG I. BR., GERMANY

Email address: bartels@mathematik.uni-freiburg.de

TEXAS A& M UNIVERSITY, COLLEGE STATION, TX 77843, USA

Email address: bonito@tamu.edu

FAKULTÄT MATHEMATIK, TECHNISCHE UNIVERSITÄT DRESDEN, ZELLESCHER WEG
12–14, 01069 DRESDEN, GERMANY

Email address: peter.hornung@tu-dresden.de

DEPARTMENT OF MATHEMATICS AND STATISTICS, PORTLAND STATE UNIVERSITY,
PORTLAND OR 97201, USA

Email address: mneunteu@pdx.edu